

A ZERO-ONE LAW FOR IMPROVEMENTS TO DIRICHLET'S THEOREM

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ABSTRACT. We give an integrability condition on a function ψ guaranteeing that for almost all (or almost no) $x \in \mathbb{R}$, the system $|qx - p| \leq \psi(t)$, $|q| < t$ is solvable in $p \in \mathbb{Z}$, $q \in \mathbb{Z} \setminus \{0\}$ for sufficiently large t . Along the way, we characterize such x in terms of the growth of their continued fraction entries, and we establish that Dirichlet's Approximation Theorem is sharp in a very strong sense. Higher-dimensional generalizations are discussed at the end of the paper.

1. INTRODUCTION AND MOTIVATION

The starting point for the present paper, as well as for numerous endeavors in the theory of Diophantine approximation, is the following theorem, established by Dirichlet in 1842:

Theorem 1.1 (Dirichlet's Theorem). *For any $x \in \mathbb{R}$ and $t \geq 1$ there exist $q \in \mathbb{Z} \setminus \{0\}$, $p \in \mathbb{Z}$ such that*

$$|qx - p| \leq \frac{1}{t} \quad \text{and} \quad |q| < t. \quad (1.1)$$

In other words, one of the first $t - 1$ multiples of any real number is $\frac{1}{t}$ -close to an integer. See e.g. [Ca1, Theorem I.I] or [Sch, Theorem I.1A]. In many cases the above theorem has been applied through its corollary, also exhibited by Dirichlet:

Corollary 1.2 (Dirichlet's Corollary). *For any $x \in \mathbb{R}$ there exist infinitely many $q \in \mathbb{Z}$ such that*

$$|qx - p| < \frac{1}{|q|} \quad \text{for some } p \in \mathbb{Z}. \quad (1.2)$$

The two statements above give a rate of approximation which works for all x and serve as a beginning of the *metric theory of Diophantine approximation*, which is concerned with understanding sets of x satisfying conclusions similar to those of Theorem 1.1 and Corollary 1.2 with the right hand sides of (1.1) and (1.2) replaced by faster decaying functions of t and $|q|$ respectively. Those sets are very well studied in the setting of Corollary 1.2. Indeed, for a function $\psi : [t_0, \infty) \rightarrow \mathbb{R}_+$, where $t_0 \geq 1$ is fixed, let us define $A(\psi)$, the set of ψ -*approximable* real numbers, to be the set of $x \in \mathbb{R}$ for which there exist infinitely many $q \in \mathbb{Z}$ such that

$$|qx - p| \leq \psi(|q|) \quad \text{for some } p \in \mathbb{Z}. \quad (1.3)$$

In what follows we will use the notation $\psi_1(t) = 1/t$. Then Corollary 1.2 asserts that $A(\psi_1) = \mathbb{R}$. It is well known that there exists $c > 0$ such that $A(c\psi_1) \neq \mathbb{R}$; more precisely, numbers x which do not belong to $A(c\psi_1)$ for some $c > 0$ are called *badly approximable*. This is equivalent to their continued fraction coefficients being uniformly bounded. It is known

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that those x form a set of full Hausdorff dimension [J]. However, the Lebesgue measure of the set of badly approximable numbers is zero; in other words, $A(c\psi_1)$ is co-null for any $c > 0$. Precise conditions for the Lebesgue measure of $A(\psi)$ to be zero or full are given by

Theorem 1.3 (Khinchine's Theorem). *Given a non-increasing ψ , the set $A(\psi)$ has zero (resp. full) measure if and only if the series $\sum_k \psi(k)$ converges (resp. diverges).*

Quite surprisingly, to the best of our knowledge no such clean statement has yet been proved in the set-up of Theorem 1.1. This is the aim of the present paper.

We start by introducing the following definition: for ψ as above, let $D(\psi)$ be the set of ψ -Dirichlet numbers, that is, $x \in \mathbb{R}$ for which the system

$$|qx - p| \leq \psi(t) \quad \text{and} \quad |q| < t \quad (1.4)$$

has a nontrivial integer solution for all large enough t . In other words, we have replaced t^{-1} in (1.1) with $\psi(t)$, demanding the existence of nontrivial integer solutions for all t except those belonging to a bounded set. Theorem 1.1 asserts that $D(\psi_1) = \mathbb{R}$, and it is easy to see that $D(\psi) \subset A(\psi)$ whenever ψ is eventually non-increasing. However the two sets differ significantly for functions ψ decaying faster than ψ_1 . For example, it has been observed by Davenport and Schmidt [DS1] that the set $D(c\psi_1)$ of $c\psi_1$ -Dirichlet numbers has Lebesgue measure zero for any $c < 1$. Moreover, they showed [DS1, Theorem 1] that a real number belongs to $D(c\psi_1)$ for some $c < 1$ if and only if it is irrational and badly approximable. These are precisely those irrational numbers whose continued fraction expansion is infinite and has uniformly bounded coefficients. This naturally motivates the following questions:

Question 1.4. *Given a non-increasing function ψ , can one characterize $x \in D(\psi)$ in terms of its continued fraction expansion?*

Question 1.5. *Is Dirichlet's theorem sharp in the sense that if ψ is non-increasing and $\psi(t) < \psi_1(t)$ for sufficiently large t , then there exists $x \in \mathbb{R}$ which is not ψ -Dirichlet?*

Question 1.6. *What is a necessary and sufficient condition on a non-increasing function ψ (presumably, expressed in the form of convergence/divergence of a certain series) guaranteeing that the set $D(\psi)$ has zero or full measure?*

In this paper we answer Questions 1.4 and 1.5 in the affirmative and give an answer to Question 1.6 under an additional assumption that the function $t \mapsto t\psi(t)$ is non-decreasing. Specifically, we will prove the following

Theorem 1.7. *If $\psi : [t_0, \infty) \rightarrow \mathbb{R}_+$ is non-increasing and $\psi(t) < \psi_1(t)$ for sufficiently large t , then $D(\psi) \neq \mathbb{R}$.*

Theorem 1.8. *Let $\psi : [t_0, \infty) \rightarrow \mathbb{R}_+$ be non-increasing such that the function $t \mapsto t\psi(t)$ is non-decreasing and*

$$t\psi(t) < 1 \quad \text{for all } t \geq t_0. \quad (1.5)$$

Then if

$$\int_{t_0}^{\infty} \frac{-\log(1 - t\psi(t))(1 - t\psi(t))}{t} dt = \infty \quad (\text{resp. } < \infty) \quad (1.6)$$

then the Lebesgue measure of $D(\psi)$ (resp. of $D(\psi)^c$) is zero.

We note that (1.5) is a natural assumption: if it is not satisfied, then $D(\psi) = \mathbb{R}$ in view of Theorem 1.1. As an example, taking $\psi = c\psi_1$ in Theorem 1.8 makes the integral in (1.6), with $t_0 = 1$, look like

$$\int_1^\infty \frac{-c \log(1-c)}{t} dt,$$

thereby recovering the aforementioned result of Davenport and Schmidt stating that $D(c\psi_1)$ has measure zero for $c < 1$. Here are two more examples¹:

- If $\psi(t) = \frac{1-at^{-k}}{t}$ for $k \geq 0$, $a > 0$, then the integral in (1.6) is proportional to

$$\int_1^\infty \frac{-\log(t^{-k})t^{-k}}{t} dt = \int_1^\infty \frac{k \log t}{t^{k+1}} dt.$$

Thus $D(\psi)$ has full measure whenever $k > 0$.

- If $\psi(t) = \frac{1-a(\log t)^{-k}}{t}$ for $k \geq 0$, $a > 0$, we are led to consider

$$\int_e^\infty \frac{k \log \log t}{t(\log t)^k} dt = \int_1^\infty \frac{k \log u}{u^k} du.$$

In this case $D(\psi)$ has full measure if $k > 1$ and zero measure otherwise.

The structure of the paper is as follows. Theorem 1.7 is proved in the next section, following some lemmas expressing the notion of Dirichlet improvability via continued fractions. In §3 we discuss dynamics of the Gauss map $x \mapsto \frac{1}{x} - \lfloor \frac{1}{x} \rfloor$ in the unit interval and, following [Ph], establish a dynamical Borel-Cantelli lemma for families of subsets of a particular form, which is used to prove Theorem 1.8. In the last section of the paper we briefly discuss possible higher-dimensional generalizations.

2. CONTINUED FRACTIONS

Denote by $\langle x \rangle$ the distance from x to the nearest integer. Throughout the sequel, $a_n = a_n(x)$ ($n = 1, 2, \dots$) will denote the n th entry in the continued fraction expansion of $x \in [0, 1)$. $q_n = q_n(x)$ will refer to the denominator of the n th convergent to x . That is

$$[a_1(x), a_2(x), \dots, a_n(x)] = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\dots + \frac{1}{a_n}}}} = \frac{p_n}{q_n}$$

with p_n, q_n coprime. If we take $q_0 = 1$, $\{q_n\}_{n=0}^\infty$ may be defined as the increasing sequence of positive integers with the property $\langle q_n x \rangle < \langle qx \rangle$ for all positive integers $q < q_n$. The sequences $\{a_n\}$, $\{q_n\}$ are closely related. We refer the reader to [Kh] or the first chapter of [Ca] for background on the theory of continued fractions².

We prefer to work with x for which the sequences $q_n(x)$, $a_n(x)$ do not terminate; that is, exclude the case $x \in \mathbb{Q}$. Since all the properties that concern us are invariant under translation by \mathbb{Z} , we will only consider $x \in [0, 1) \setminus \mathbb{Q}$.

Lemma 2.1. *Let $\psi : [t_0, \infty) \rightarrow \mathbb{R}_+$ be non-increasing. Then $x \in [0, 1] \setminus \mathbb{Q}$ is ψ -Dirichlet if and only if $\langle q_{n-1}x \rangle \leq \psi(q_n)$ for sufficiently large n .*

¹The functions below are not non-increasing, but *eventually* decreasing; clearly only the eventual behavior of ψ is relevant.

²Note however that Cassels definition of the sequence $\{q_n\}$ differs from that of Khintchine by one index. That is, Cassels has $q_1 = 1, q_2 = a_1, \dots$. We have adopted Khintchine's notation.

Proof. Suppose $x \in [0, 1] \setminus \mathbb{Q}$ is ψ -Dirichlet. Then for sufficiently large n there exists a positive integer, q , with $\langle qx \rangle \leq \psi(q_n)$, $q < q_n$. Since $\langle q_{n-1}x \rangle \leq \langle qx \rangle$ whenever $q < q_n$, we have $\langle q_{n-1}x \rangle \leq \psi(q_n)$ for sufficiently large n . Conversely, suppose $\langle q_{n-1}x \rangle \leq \psi(q_n)$ for $n \geq N$. Then for a real number $t > q_N$, write $q_{n-1} < t \leq q_n$. Then $\langle q_{n-1}x \rangle \leq \psi(t)$ since ψ is non-increasing. Thus x is ψ -Dirichlet. \square

Lemma 2.1 is one step toward rephrasing the ψ -Dirichlet property of x in terms of the growth of the continued fraction entries, $a_n(x)$. In proving Theorem 1.8, we will ultimately weaken our condition for $x \in D(\psi)$ to an almost-everywhere statement. But before we do so, we give another condition for the ψ -Dirichlet property which holds for all x , allowing us to answer Question 1.5.

For fixed $x = [a_1, a_2, \dots]$, consider the sequences $\theta_{n+1} = [a_{n+1}, a_n, \dots]$, $\phi_n = [a_n, a_{n-1}, \dots, a_1]$. These are related to the sequences q_n , $\langle q_{n-1}x \rangle$ via

$$(1 + \theta_{n+1}\phi_n)^{-1} = q_n \langle q_{n-1}x \rangle, \quad (2.1)$$

see [Ca1, §II.2]³. This is our device for passing from Lemma 2.1 to continued fractions.

Lemma 2.2. *Let $x \in [0, 1] \setminus \mathbb{Q}$, and let $\psi : [t_0, \infty) \rightarrow \mathbb{R}_+$ be non-increasing. Then*

- (i) *x is ψ -Dirichlet if $a_{n+1}a_n \leq \frac{1}{4} \left((q_n\psi(q_n))^{-1} - 1 \right)^{-1}$ for all sufficiently large n .*
- (ii) *x is not ψ -Dirichlet if $a_{n+1}a_n > \left((q_n\psi(q_n))^{-1} - 1 \right)^{-1}$ for infinitely many n .*

Proof. Fix $x \in [0, 1] \setminus \mathbb{Q}$. Using (2.1), Lemma 2.1 becomes

$$x \in D(\psi) \text{ if and only if } (1 + \theta_{n+1}\phi_n)^{-1} \leq q_n\psi(q_n) \text{ for all large } n. \quad (2.2)$$

Since $(a_{n+1} + \frac{1}{a_{n+2}})(a_n + \frac{1}{a_{n-1}}) \leq 4a_{n+1}a_n$, we have

$$\begin{aligned} \left(1 + \frac{1}{a_{n+1}} \cdot \frac{1}{a_n}\right)^{-1} &\leq (1 + \theta_{n+1}\phi_n)^{-1} \\ &\leq \left(1 + \frac{1}{a_{n+1} + \frac{1}{a_{n+2}}} \cdot \frac{1}{a_n + \frac{1}{a_{n-1}}}\right)^{-1} \leq \left(1 + \frac{1}{4a_{n+1}a_n}\right)^{-1} \end{aligned}$$

Hence from (2.2), $x \in D(\psi)$ if

$$\left(1 + \frac{1}{4a_{n+1}a_n}\right)^{-1} \leq q_n\psi(q_n)$$

for sufficiently large n . We get the first assertion of the lemma by solving for a_na_{n+1} . Similarly, $x \notin D(\psi)$ if

$$\left(1 + \frac{1}{a_{n+1}a_n}\right)^{-1} > q_n\psi(q_n)$$

for unbounded n . Solving for a_na_{n+1} gives the second assertion of the lemma. \square

Remark 2.3. *Lemma 2.2 generalizes the result of Davenport and Schmidt mentioned in the introduction [DS1, Theorem 1] stating that $x \in D(c\psi_1)$ for some $c < 1$ if and only if the sequence $a_n(x)$ is uniformly bounded.*

³In truth, Cassels' formula reads $(1 + \theta_{n+1}\phi_n)^{-1} = q_{n+1}\langle q_nx \rangle$ because his q_n 's are shifted by one index, as we have already noted.

To answer Question 1.5 we recall the recurrence

$$q_n = a_n q_{n-1} + q_{n-2}, \quad (2.3)$$

see [Kh, Theorem 1].

Theorem 1.7. *If $\psi : [t_0, \infty) \rightarrow \mathbb{R}_+$ is non-increasing and $\psi(t) < \psi_1(t)$ for sufficiently large t , then $D(\psi) \neq \mathbb{R}$.*

Proof. By the recurrence (2.3), q_n depends only on a_1, \dots, a_n . Since $t\psi(t) < 1$, we may construct $x = [a_1, a_2, \dots]$ by successively choosing a_{n+1} so that part (ii) of Lemma 2.2 is satisfied. \square

Remark 2.4. *We point out that for a given ψ , the proof of Theorem 1.7 is entirely constructive, since each q_n is determined recursively by the preceding choice of a_n . Also note that the proof constructs x such that the system (1.4) is insoluble when $t = q_n$ for all sufficiently large n – not just for infinitely many q_n .*

The next lemma will give a condition for Dirichlet improvability depending *only* on the continued fraction entries (i.e. removing the dependence on q_n seen in Lemma 2.2), at the price of a set of measure zero and an added hypothesis that the function $t \mapsto t\psi(t)$ is non-decreasing.

Lemma 2.5. *There exist $b, B > 0$ such that for almost all x and for any $\psi : [t_0, \infty) \rightarrow (0, 1)$ which is non-increasing and for which the function $t \mapsto t\psi(t)$ is non-decreasing, the following holds:*

- (i) *x is ψ -Dirichlet if $a_{n+1}a_n \leq \frac{1}{4} \cdot ([b^n\psi(b^n)]^{-1} - 1)^{-1}$ for all sufficiently large n .*
- (ii) *x is not ψ -Dirichlet if $a_{n+1}a_n > ([B^n\psi(B^n)]^{-1} - 1)^{-1}$ for infinitely many n .*

Proof. There exists $b > 0$ such that for almost every x , $b^n \leq q_n$ for all large n [Kh, §4]. There also exists $B > 0$ so that for almost all x , $q_n \leq B^n$ for all large n [Kh, §14]. For such x , parts (i) and (ii) follow immediately from Lemma 2.2 and the monotonicity of $t \mapsto t\psi(t)$ \square

3. BOREL-CANTELLI LEMMAS

For almost every x , we have reduced the ψ -Dirichlet property of x to the growth of its continued fraction entries. The *Gauss map*,

$$T : [0, 1] \setminus \mathbb{Q} \rightarrow [0, 1] \setminus \mathbb{Q}, \quad x \mapsto x^{-1} - \lfloor x^{-1} \rfloor \quad (3.1)$$

has the convenient property $T([a_1, a_2, a_3, \dots]) = [a_2, a_3, a_4, \dots]$, and it preserves the *Gauss measure*,

$$\mu(A) = \frac{1}{\log 2} \int_A \frac{1}{1+x} dx \quad (3.2)$$

We will use two results of Philipp [Ph] related to the mixing rate of T and the divergence case of the Borel-Cantelli Lemma.

Theorem 3.1. [Ph, Theorem 2.3]. *Let E_n , $n \geq 1$, be a sequence of measurable sets in an arbitrary probability space (X, ν) . Denote by $A(N, x)$ the number of integers $n \leq N$ such that $x \in E_n$. Put*

$$\phi(N) = \sum_{n \leq N} \nu(E_n)$$

Suppose that there exists a convergent series $\sum_{j \geq 1} C_j$ with $C_j \geq 0$ such that for all integers $m > n$ we have

$$\nu(E_n \cap E_m) \leq \nu(E_n)\nu(E_m) + \nu(E_m)C_{m-n}. \quad (3.3)$$

Then for any $\epsilon > 0$ one has

$$A(N, x) = \phi(N) + O_\epsilon(\phi^{1/2}(N) \log^{3/2+\epsilon} \phi(N))$$

for almost all x .

Remark 3.2. Since $|\nu(E_n \cap E_m) - \nu(E_n)\nu(E_m)| \leq 2\nu(E_m)$, Theorem 3.1 can be trivially strengthened: given any $\ell > 0$, the conclusion of the theorem holds provided the inequality (3.3) holds whenever $m > n + \ell$.

Theorem 3.3. [Ph, Theorem 3.2] There exist $c_0 > 0$ and $0 < \gamma < 1$ with the following property. For any $k \in \mathbb{N}$, let

$$E = \{x \in [0, 1] \setminus \mathbb{Q} : a_1(x) = r_1, a_2(x) = r_2, \dots, a_k(x) = r_k\}$$

for some $r_1, \dots, r_k \in \mathbb{N}$, and let $F \subset [0, 1]$ be measurable. Then

$$|\mu(E \cap T^{-n-k}F) - \mu(E)\mu(F)| \leq c_0\mu(E)\mu(F)\gamma^{\sqrt{n}}. \quad (3.4)$$

Corollary 3.4. Let c_0 and γ be as in Theorem 3.3. Fix $k \in \mathbb{N}$. If $E = \cup_m E_m$ is a countable disjoint union of sets of the form

$$E_m = \{x \in [0, 1] \setminus \mathbb{Q} : a_1(x) = r_1, a_2(x) = r_2, \dots, a_k(x) = r_k\},$$

and $F \subset [0, 1]$ is measurable, then (3.4) holds for all $n \in \mathbb{N}$.

Proof. Theorem 3.3 gives the result for $E = E_m$. In general we have

$$\begin{aligned} |\mu(\cup_m E_m \cap T^{-n-k}F) - \mu(\cup_m E_m)\mu(F)| &= \left| \sum_m \mu(E_m \cap T^{-n-k}F) - \mu(E_m)\mu(F) \right| \\ &\leq \sum_m c_0\mu(E_m)\mu(F)\gamma^{\sqrt{n}} = c_0\mu(E)\mu(F)\gamma^{\sqrt{n}}. \end{aligned}$$

□

Lemma 3.5. Suppose A_n ($n \in \mathbb{N}$) is a sequence of sets such that each A_n is a countable disjoint union of sets of the form

$$J_{c,d} := \{x \in [0, 1] \setminus \mathbb{Q} : a_1(x) = c, a_2(x) = d\}.$$

If $\sum_n \mu(A_n) = \infty$ (resp. $< \infty$), then for almost every (resp. almost no) $x \in [0, 1]$ one has $T^n(x) \in A_n$ for infinitely many n .

Proof. The convergence case follows from the Borel-Cantelli Lemma and the fact that μ is T -invariant. For the divergence case, for $m > n + 2$ write

$$\begin{aligned} \mu(T^{-n}A_n \cap T^{-m}A_m) &= \mu(A_n \cap T^{-(m-n)}A_m) \leq \mu(A_n)\mu(A_m) + c_0\mu(A_m)\mu(A_n)\gamma^{\sqrt{m-n-2}} \\ &\leq \mu(A_n)\mu(A_m) + \mu(A_m)c_0\gamma^{\sqrt{m-n-2}} = \mu(T^{-n}A_n)\mu(T^{-m}A_m) + \mu(T^{-m}A_m)c_0\gamma^{\sqrt{m-n-2}} \end{aligned}$$

for c_0, γ as in Theorem 3.3. The sets $T^{-n}A_n$ therefore satisfy the condition of Theorem 3.1 (in light of Remark 3.2). By that theorem, $\sum_n \mu(T^{-n}A_n) = \infty$ guarantees that almost all x lie in $T^{-n}A_n$ for infinitely many n . □

Lemma 3.6. *Let $\beta : \mathbb{N} \rightarrow [1, \infty)$ be any function with $\lim_{n \rightarrow \infty} \beta(n) = \infty$. If*

$$\sum_n \frac{\log(\beta(n))}{\beta(n)} < \infty \text{ (resp. } = \infty),$$

then almost every (resp. almost no) $x \in [0, 1] \setminus \mathbb{Q}$ has $a_{n+1}(x)a_n(x) \leq \beta(n)$ for sufficiently large n .

Proof. We may assume β is integer-valued. Define

$$A_n := \{x : a_1(x)a_2(x) > \beta(n)\} = \bigcup_{a=1}^{\infty} \bigcup_{b=\lfloor \frac{\beta(n)}{a} + 1 \rfloor}^{\infty} \left(\frac{1}{a + \frac{1}{b}}, \frac{1}{a + \frac{1}{b+1}} \right) = \bigcup_{a=1}^{\infty} \left(\frac{1}{a + \frac{1}{\lfloor \frac{\beta(n)}{a} + 1 \rfloor}}, \frac{1}{a} \right)$$

Clearly $x \in [0, 1] \setminus \mathbb{Q}$ has $a_{n+1}a_n > \beta(n)$ if and only if $T^{n-1}(x) \in A_n$, where T as in (3.1). By Lemma 3.5, it suffices to show

$$c^{-1}\mu(A_n) \leq \frac{\log \beta(n)}{\beta(n)} \leq c\mu(A_n)$$

for some $c > 0$ for all large n . In fact, since $\frac{\lambda}{\log 2} \leq \mu \leq \frac{2\lambda}{\log 2}$, where λ is Lebesgue measure on $[0, 1]$, it suffices to show

$$c^{-1}\lambda(A_n) \leq \frac{\log \beta(n)}{\beta(n)} \leq c\lambda(A_n).$$

We have

$$\begin{aligned} A_n &\subset \left(\bigcup_{a \leq \beta(n)} \left(\frac{1}{a + \frac{a}{\beta(n)}}, \frac{1}{a} \right) \right) \cup \left(\bigcup_{a > \beta(n)} \left(\frac{1}{a+1}, \frac{1}{a} \right) \right) \subset \\ &\quad \left(\bigcup_{a \leq \beta(n)} \left(\frac{1}{a + \frac{a}{\beta(n)}}, \frac{1}{a} \right) \right) \cup \left(0, \frac{1}{\beta(n)} \right) = \\ &\quad \left(\frac{1}{1 + \frac{1}{\beta(n)}}, 1 \right) \cup \left(\bigcup_{a=2}^{\beta(n)} \left(\frac{1}{a + \frac{a}{\beta(n)}}, \frac{1}{a} \right) \right) \cup \left(0, \frac{1}{\beta(n)} \right) \end{aligned}$$

So

$$\begin{aligned} \lambda(A_n) &\leq 1 - \frac{1}{1 + \frac{1}{\beta(n)}} + \int_1^{\beta(n)} \frac{1}{a} - \frac{1}{a + \frac{a}{\beta(n)}} da + \frac{1}{\beta(n)} \\ &= 1 - \frac{1}{1 + \frac{1}{\beta}} + \log \beta \left(1 - \frac{1}{1 + \frac{1}{\beta}} \right) + \frac{1}{\beta} = \frac{1}{\beta + 1} + \frac{\log \beta}{1 + \beta} + \frac{1}{\beta} \asymp \frac{\log \beta(n)}{\beta(n)} \end{aligned}$$

To see the asymptotic lower bound, we start with

$$A_n \supset \bigcup_{a=1}^{\beta(n)} \left(\frac{1}{a + \frac{1}{\frac{\beta(n)}{a} + 1}}, \frac{1}{a} \right).$$

Then

$$\begin{aligned}
\lambda(A_n) &\geq \int_1^\beta \frac{1}{a} - \frac{1}{a + \frac{1}{\frac{\beta}{a} + 1}} da = \int_1^\beta \frac{1}{a(a + \beta + 1)} da \\
&= \int_1^\beta \frac{(\beta + 1)^{-1}}{a} - \frac{(\beta + 1)^{-1}}{\beta + a + 1} da = \frac{\log(\beta)}{\beta + 1} + \frac{\log(\frac{\beta+2}{2\beta+1})}{\beta + 1} \asymp \frac{\log(\beta)}{\beta}.
\end{aligned}$$

□

We are now ready to characterize ψ such that $D(\psi)$ has zero/full measure.

Theorem 1.8. *Let $\psi : [t_0, \infty) \rightarrow (0, 1)$ be non-increasing such that the function $t \mapsto t\psi(t)$ is non-decreasing and (1.5) holds. Then if*

$$\int_{t_0}^\infty \frac{-\log(1 - t\psi(t))(1 - t\psi(t))}{t} dt = \infty \quad (\text{resp. } < \infty)$$

then the Lebesgue measure of $D(\psi)$ (resp. of $D(\psi)^c$) is zero.

Proof. For ease of notation, let us write $\varphi(t) := t\psi(t)$. We may assume $\varphi \rightarrow 1$ since $D(c\psi_1)$ is known to have measure zero for all $c < 1$. Suppose

$$\int_{t_0}^\infty \frac{-\log(1 - \varphi(t))(1 - \varphi(t))}{t} dt < \infty,$$

and suppose b is the constant from Lemma 2.5. Then

$$\int_{\log_b t_0}^\infty -\log(1 - \varphi(b^t))(1 - \varphi(b^t)) dt < \infty.$$

Since φ is non-decreasing and $\varphi \rightarrow 1$, differentiating $x \mapsto -\log(1 - x)(1 - x)$ shows that the integrand is eventually non-increasing. Therefore

$$\begin{aligned}
\int_{\log_b t_0}^\infty -\log(1 - \varphi(b^t))(1 - \varphi(b^t)) dt < \infty &\implies \\
\sum_n -\log(1 - \varphi(b^n))(1 - \varphi(b^n)) < \infty &\implies \sum_n \frac{-\log(1 - \varphi(b^n))}{[\frac{1}{\varphi(b^n)} - 1]^{-1}} < \infty \implies \\
\sum_n \frac{-\log(1 - \varphi(b^n)) + \log(\varphi(b^n))}{[\frac{1}{\varphi(b^n)} - 1]^{-1}} < \infty &\implies \sum_n \frac{-\log(\frac{1}{\varphi(b^n)} - 1)}{[\frac{1}{\varphi(b^n)} - 1]^{-1}} < \infty \implies \\
\sum_n \frac{\log([\frac{1}{\varphi(b^n)} - 1]^{-1})}{[\frac{1}{\varphi(b^n)} - 1]^{-1}} < \infty &\implies \sum_n \frac{\log\left(\frac{1}{4}[\frac{1}{\varphi(b^n)} - 1]^{-1}\right)}{\frac{1}{4}[\frac{1}{\varphi(b^n)} - 1]^{-1}} < \infty,
\end{aligned}$$

where in the second and third implications we have used $\varphi \rightarrow 1$. By Lemmas 2.5 and 3.6, almost every $x \in \mathbb{R}$ is ψ -Dirichlet. The divergence case is proved the same way: change all the “<” signs to “=”, and change b to B (from Lemma 2.5) in the preceding argument. □

4. GENERALIZATIONS TO HIGHER DIMENSIONS

Let m, n be positive integers, and denote by $M_{m,n}$ the space of $m \times n$ matrices with real entries. The following is the general form of Dirichlet's Theorem on simultaneous Diophantine approximation, see e.g. [Ca1, §I.5] or [Sch, Theorem II.1E]:

Theorem 4.1. *For any $Y \in M_{m,n}$ (viewed as a system of m linear forms in n variables, $\mathbf{x} \mapsto Y\mathbf{x}$) and for any $t \geq 1$ there exist $\mathbf{q} = (q_1, \dots, q_n) \in \mathbb{Z}^n \setminus \{0\}$ and $\mathbf{p} = (p_1, \dots, p_m) \in \mathbb{Z}^m$ satisfying the following system of inequalities:*

$$\|Y\mathbf{q} - \mathbf{p}\| \leq t^{-n/m} \quad \text{and} \quad \|\mathbf{q}\| < t. \quad (4.1)$$

Here $\|\cdot\|$ stands for the norm on \mathbb{R}^k given by $\|\mathbf{x}\| = \max_{1 \leq i \leq k} |x_i|$.

Similarly⁴ to what was done for $m = n = 1$, given a non-increasing $\psi : [t_0, \infty) \rightarrow \mathbb{R}_+$, let us say that $Y \in M_{m,n}$ is ψ -Dirichlet, and write $Y \in D_{m,n}(\psi)$, if for every sufficiently large t one can find $\mathbf{q} \in \mathbb{Z}^n \setminus \{0\}$ and $\mathbf{p} \in \mathbb{Z}^m$ with

$$\|Y\mathbf{q} - \mathbf{p}\|^m \leq \psi(t) \quad \text{and} \quad \|\mathbf{q}\|^n \leq t. \quad (4.2)$$

This way Theorem 4.1 implies that $D_{m,n}(\psi_1) = M_{m,n}$. Furthermore, the sharpness result of Davenport and Schmidt mentioned in the introduction also holds in higher dimensions: the Lebesgue measure of $D_{m,n}(c\psi_1)$ is zero for any $c < 1$. See [DS2, Theorem 1] for the case $\min(m, n) = 1$, and [KWe, Theorem 4] for further generalizations. This naturally motivates higher-dimensional analogues of Questions 1.5 and 1.6:

Question 4.2. *Is Theorem 4.1 sharp in the sense that if ψ is non-increasing and $\psi(t) < \psi_1(t)$ for sufficiently large t , then there exists $Y \in M_{m,n}$ which is not ψ -Dirichlet?*

Question 4.3. *For fixed $m, n \in \mathbb{N}$, what is a necessary and sufficient condition on a non-increasing ψ (presumably, expressed in the form of convergence/divergence of a certain series) guaranteeing that the set $D_{m,n}(\psi)$ has zero/full measure?*

Of course, in higher dimensions the machinery of continued fractions is no longer available. It is nonetheless still possible to restate the problem in terms of a shrinking target phenomenon in a dynamical system, as we have done in the one-dimensional case. This approach is based on ideas from [DS2] and [Da], and, in a more explicit form – on [KM, §8], where the Khintchine-Groshev theorem (the natural higher dimensional analogue of Theorem 1.3) is proved using a dynamical Borel-Cantelli Lemma for a diagonal flow on the space of unimodular lattices in \mathbb{R}^{m+n} . The starting point for the reduction is the “Dani Correspondence”:

Lemma 4.4. [KM, Lemma 8.3] *Fix $m, n \in \mathbb{N}$ and $t_0 \geq 1$, and let $\psi : [t_0, \infty) \rightarrow \mathbb{R}_+$ be a continuous, non-increasing function. Then there exists a unique continuous function*

$$r : [s_0, \infty) \rightarrow \mathbb{R}, \quad \text{where } s_0 = \frac{m}{m+n} \log t_0 - \frac{m}{m+n} \log \psi(t_0),$$

such that the function $s \mapsto s - nr(s)$ is strictly increasing and unbounded, the function $s \mapsto s + mr(s)$ is nondecreasing, and

$$\psi(e^{s-nr(s)}) = e^{-s-mr(s)} \quad \text{for all } s \geq s_0.$$

⁴A reader will notice a minor difference in definitions of $D(\psi)$ and $D_{1,1}(\psi)$, namely the replacement of $<$ in the second inequality in (1.4) with \leq . It is not hard to show that under an additional assumption of the continuity of ψ , Theorem 1.8, with the same integrability condition (1.6), holds for $D_{1,1}(\psi)$ in place of $D(\psi)$.

Denote by X the space of unimodular lattices in \mathbb{R}^{m+n} , and define

$$\Delta : X \rightarrow \mathbb{R}, \Lambda \mapsto -\log \inf_{\mathbf{v} \in \Lambda \setminus \{0\}} \|\mathbf{v}\|.$$

$X \cong \mathrm{SL}_{m+n}(\mathbb{R})/\mathrm{SL}_{m+n}(\mathbb{Z})$ is a noncompact homogeneous space, and according to Mahler's Compactness Criterion, a subset K of X is relatively compact if and only if the restriction of Δ to K is bounded from above. Also, in view of Minkowski's Lemma, Δ is always bounded from below by 0. Furthermore,

$$K_0 := \Delta^{-1}(0) \tag{4.3}$$

is a union of finitely many compact submanifolds of X , whose structure is explicitly described by Hajós-Minkowski Theorem, see [Ca2, §XI.1.3] or [Sh, Theorem 2.3].

For $Y \in M_{m,n}$, define

$$\Lambda_Y := \begin{pmatrix} I_m & Y \\ 0 & I_n \end{pmatrix} \mathbb{Z}^{m+n} \in X.$$

Finally, define

$$g_s := \mathrm{diag}(e^{s/m}, \dots, e^{s/m}, e^{-s/n}, \dots, e^{-s/n}),$$

where there are m copies of $e^{s/m}$ and n copies of $e^{-s/n}$. We may now rephrase the ψ -Dirichlet property of $Y \in M_{m,n}$ as a statement about the orbit of Λ_Y in the dynamical system (X, g_s) :

Proposition 4.5. *Fix positive integers m, n , and let $\psi : [t_0, \infty) \rightarrow (0, 1]$ be continuous, non-increasing, and not identically 1. Let $r = r_\psi$ be as in Lemma 4.4. Then $Y \in D_{m,n}(\psi)$ if and only if*

$$\Delta(g_s \Lambda_Y) \geq r_\psi(s)$$

for sufficiently large s

Proof. Recall that $Y \in D_{m,n}(\psi)$ if and only if for large enough t the system (4.2) has a solution (\mathbf{p}, \mathbf{q}) with $\mathbf{q} \in \mathbb{Z}^n \setminus \{0\}$ and $\mathbf{p} \in \mathbb{Z}^m$. If $\psi(t) < 1$, all solutions $(\mathbf{p}, \mathbf{q}) \neq 0$ to this system will have $\mathbf{q} \neq 0$. Since ψ is eventually less than 1, $Y \in D_{m,n}(\psi)$ if and only if (4.2) is solvable in $(\mathbf{p}, \mathbf{q}) \in \mathbb{Z}^{m+n} \setminus \{0\}$ for sufficiently large t .

Since the function $s \mapsto s - nr(s)$ is increasing and unbounded, $Y \in D_{m,n}(\psi)$ if and only if for large enough s ,

$$\|Y\mathbf{q} - \mathbf{p}\|^m \leq \psi(e^{s-nr(s)}) = e^{-s-mr(s)} \quad \|\mathbf{q}\|^n \leq e^{s-nr(s)}$$

for some $(\mathbf{p}, \mathbf{q}) \in \mathbb{Z}^{m+n} \setminus \{0\}$. This is equivalent to

$$e^{s/m} \|Y\mathbf{q} - \mathbf{p}\| \leq e^{-r(s)} \quad e^{-s/n} \|\mathbf{q}\| \leq e^{-r(s)},$$

which is the same as $\Delta(g_s \Lambda_Y) \geq r_\psi(s)$. □

Therefore $Y \notin D_{m,n}(\psi)$ if and only if $g_s \Lambda_Y \in \Delta^{-1}([0, r_\psi(s)))$ for an unbounded set of $s \in \mathbb{R}_+$. Note that the targets $\Delta^{-1}[0, r)$ are open neighborhoods of the set K_0 as in (4.3). We are thus interested in whether these shrinking targets are hit at unbounded set of times by trajectories of a measure-preserving flow.

There are some technical obstructions (perhaps surmountable) to this approach to Questions 4.2 and 4.3. However, in a forthcoming paper [KWa] we use a similar approach to solve an analogous inhomogeneous problem. Specifically, we establish a dynamical Borel-Cantelli Lemma for the flow g_s on the space of *affine* unimodular lattices in \mathbb{R}^{m+n} , and go on to prove the following result:

Theorem 4.6. *Let $\psi : [1, \infty) \rightarrow (0, 1]$ be a non-increasing continuous function. If*

$$\int_1^\infty t^{-2}\psi(t)^{-1} dt < \infty \quad (\text{resp. } = \infty),$$

then for almost all (resp. almost no) pairs $Y \in M_{m \times n}$, $\mathbf{b} \in \mathbb{R}^m$, the system

$$\|Y\mathbf{q} + \mathbf{b} - \mathbf{p}\| \leq \psi(t) \quad \|\mathbf{q}\| < t$$

is solvable in integer vectors $\mathbf{q} \in \mathbb{Z}^n \setminus \{0\}$ and $\mathbf{p} \in \mathbb{Z}^m$ for sufficiently large t .

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